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Topological classification of non-degenerate quadratic system

Aleksandr Voldman

TOPOLOGICAL CLASSIFICATION OF NON-DEGENERATE QUADRATIC SYSTEM

A Project
Presented to the
Faculty of
California State University,
San Bernardino

In Partial Fulfillment
of the Requirements for the Degree
Master of Arts
in
Mathematics

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ABSTRACT

It is known that the complete qualitative investigation of polynomial systems of differential equations consists of the partition of its coefficient space into disjoint subsets, so that any two systems in the same subset are topologically equivalent. Topological equivalence between two systems is given, as usual, by a global homeomorphism which sends the orbits of the system one to the orbits of the system two. Obtained conditions in the coefficients, which distinguish various topological classes are affine invariant. They yield the required partition of R^{12}. The determined necessary and sufficient conditions for distinguishing the phase portraits of a quadratic system are affine invariant and algebraic, i.e. they are expressed as equalities or inequalities between the polynomials of the coefficients. Thus, the qualitative investigation of such quadratic system have been reduced to arithmetic operations.

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Introduction

Consider the following two-dimensional autonomous system of differential equations

$$\frac{dx^1}{dt} = P^{(1)}(x^1, x^2), \quad \frac{dx^2}{dt} = P^{(2)}(x^1, x^2), \tag{1.1}$$

where $P^{(j)}(x^1, x^2) \quad (j = 1, 2)$ are the real polynomials of n-degree from coordinates of the vector $x = (x^1, x^2)$

For example:

$$\frac{dx}{dt} = a + dy, \quad \frac{dy}{dt} = b + ex + fy + ny^2, \tag{$\star$}$$

Denote by R^m, $m = (n+1)(n+2)$ the space of the coefficients of the system (1.1) and let $a \in R^m$ be a point of this space. Consider the group $Q = GL(2, R)$ of linear homogenous transformations of the phase plane of system (1.1) (which is also called a group of center-affine transformations) and let us write down the linear transformation $q \in Q$ as

$$y = qx, \tag{1.2}$$

where $y = (y^1, y^2)$ is a vector of a new functions and q is 2×2 matrix.

Applying a transformation q to the system (1.1), we obtain a new system

$$\frac{dy^1}{dt} = \overline{P}^{(1)}(y^1, y^2), \quad \frac{dy^2}{dt} = \overline{P}^{(2)}(y^1, y^2), \tag{1.3}$$

the coefficients of which are also the coordinates of some point $b \in R^m$.

In our example if we apply the transformation q of the form

$$x = x_1 + x_0, \quad y = y_1 + y_0$$

to the system (1.1) we obtain

$$\frac{dx_1}{dt} = \frac{dx}{dt} = a + d(y_1 + y_0) = a + dy_0 + dy_1$$
$$\frac{dy_1}{dt} = \frac{dy}{dt} = b + e(x_1 + x_0) + f(y_1 + y_0) + n(y_1 + y_0)^2$$

that is,

$$\frac{dx_1}{dt} = \frac{dx}{dt} = a + dy_0 + dy_1$$
$$\frac{dy_1}{dt} = \frac{dy}{dt} = (b + ex_0 + fy_0 + ny_0^2) + ex_1 + (f + 2ny_0)y_1 + ny_1^2$$

Thus,

$$\frac{dx_1}{dt} = a_1 + d_1 y_1, \quad \frac{dy_1}{dt} = b_1 + e_1 x_1 + f_1 y_1 + n_1 y_1^2$$

where the new coefficients are

$$a_1 = a + dy_0, d_1 = d, b_1 = b + ex_0 + fy_0 + ny_0^2, n_1 = n$$

It is clear that the coefficients of the transformed system (1.3) depend on the coefficients of the system (1.1) and on the transformation $q \in Q$ that's why we are going to write $b = a(q)$

Definition 1.1. [1] *The polynomial $K(a, x)$ with the coefficients of the system (1.1) and the unknown variables x^1, x^2 is said to be a comitant of system (1.1) for group Q (or center-affine comitant), if there exists a function $\lambda(q)$, which depends only on the elements of the group such as*

$$K(b, y) \equiv \lambda(q) K(a, x)$$

for every $q \in Q$, $a \in R^m$ and $x = (x^1, x^2)$.

Function $\lambda(q)$ is called a **multiplier**. If $\lambda(q) \equiv 1$, then comitant $K(a, x)$ is called an **absolute**, otherwise it is a **relative**. It is known (see [1]), that

$\lambda(q) = \Delta_q^{-\kappa}$, where $\Delta_q \neq 0$ is a determinant of the linear transformation matrix and the whole number κ is called the weight of the cominant.

Definition 1.11. The weight of a polynomial affine invariant coincides with the number of factors of type ε in its tensor notation.

For example:

$I_3 = a_p^\alpha a_{\alpha q}^\beta a_{\beta\gamma}^\gamma \varepsilon^{pq}$ is of weight $\kappa = 1$ (1 - 0)

$I_{15} = a_{pr}^\alpha a_{qk}^\beta a_{\alpha s}^\gamma a_{\delta l}^\delta a_{\beta\gamma}^\mu a_{\mu\nu}^\nu \varepsilon^{pq}\varepsilon^{rs}\varepsilon^{kl}$

is of weight $\kappa = 3$ (3 - 0)

Definition 1.12. The weight of an affine comitant is equal to the number of the ε-factors with the upper indices minus the number of the ε-factors in lower indices.

For example:

$K_6 = x^\alpha x^\beta a_{\alpha\beta}^\gamma a_{\gamma\delta}^\delta$ has zero weight (0 - 0).

$K_{19} = x^\alpha a_p^\beta a_\alpha^\gamma a_{\beta q}^\delta a_{\gamma\nu}^\mu a_{\delta\mu}^\nu \varepsilon^{pq}$ has weight equal to 1 (1 - 0).

Cominant which doesn't depend on variables x^1 and x^2 is called invariant of the system (1.1) for group Q, or it is also called center-affine invariant.

Consider the following relations

$$R_i(a) = 0 \quad (1 \leq i \leq s), \tag{1.4}$$

where $R_i(a)$ are polynomials of the system (1.1).

Definition 1.2. *The polynomial relations (1.4) have an invariant meaning for group Q, if for each $q \in Q$ they are equivalent to*

$$R_i(a(q)) = 0 \quad (1 \leq i \leq s).$$

We have to admit that the main purpose for applying comitants of the system (1.1) for group Q gives

Proposition 1.1.[1] *Any system of polynomial relations (1.4) among the coefficients of the system (1.1), having invariant for group Q meaning, can be expressed through the comitants of the system (1.1) for this group.*

In order to express the conditions of the form (1.4) through comitants (this we can do according to proposition 1.1) it is neccessary to have

Definition 1.3. *A polynomial comitant $K(a, x^1, x^2)$ is called reducible, if it is expressed polynomially by comitants and invariants of lesser degrees*

Definition 1.4. *The set of polynomial comitants $\{K_\theta(a, x^1, x^2),\ \theta \in \Theta\}$ of system (1.1) under qroup Q is called a polynomial basis of comitants of this system under group Q, if any polynomial comitant $K(a, x^1, x^2)$ of system (1.1) under group Q can be exspressed in the form of a polynomial of comitants $K_\theta(a, x^1, x^2)$. Here Θ is some set of integers.*

Definition 1.5. *A polynomial basis of the comitants of system (1.1) under the group Q is called minimal, if at the removal out of it of any comitant it will cease to be a polynomial basis.*

As it follows from the Gilbert's works (as example see [2]) polynomial basis of comitants (and therefore and invariants) of system (1.1) under group Q is finite.

In [1] the method of investigating and constructing comitants of system (1.1) under group Q of center-affine transformations which based on the symbolic and classical Arnold's method has been proposed. Using this method the center-affine comitants and invariants of two-dimensional quadratic system are investigated in [1], and namely, for quadratic system:

$$\frac{dx^j}{dt} = a^j + a^j_\alpha x^\alpha + a^j_{\alpha\beta} x^\alpha x^\beta, \qquad (j, \alpha, \beta = 1, 2) \tag{1.5}$$

36 center-affine invariants were found

$$I_1 = a^\alpha_\alpha,\ I_2 = a^\alpha_\beta a^\beta_\alpha,\ I_3 = a^\alpha_p a^\beta_{\alpha q} a^\gamma_{\beta\gamma}\varepsilon^{pq},\ I_4 = a^\alpha_p a^\beta_{\beta q} a^\gamma_{\alpha\gamma}\varepsilon^{pq},$$

$$I_5 = a^\alpha_p a^\beta_{\gamma q} a^\gamma_{\alpha\beta}\varepsilon^{pq},\ I_6 = a^\alpha_p a^\beta_\gamma a^\gamma_{\alpha q} a^\delta_{\beta\delta}\varepsilon^{pq},\ I_7 = a^\alpha_{pr} a^\beta_{\alpha q} a^\gamma_{\beta s} a^\delta_{\gamma\delta}\varepsilon^{pq}\varepsilon^{rs},$$

$$I_8 = a^\alpha_{pr} a^\beta_{\alpha q} a^\gamma_{\delta s} a^\delta_{\beta\gamma}\varepsilon^{pq}\varepsilon^{rs},\quad I_9 = a^\alpha_{pr} a^\beta_{\beta q} a^\gamma_{\gamma s} a^\delta_{\alpha\delta}\varepsilon^{pq}\varepsilon^{rs},$$

$$I_{10} = a^\alpha_p a^\beta_\delta a^\gamma_\mu a^\delta_{\alpha q} a^\mu_{\beta\gamma}\varepsilon^{pq},\quad I_{11} = a^\alpha_p a^\beta_{qr} a^\gamma_{\beta s} a^\delta_{\alpha\gamma} a^\mu_{\delta\mu}\varepsilon^{pq}\varepsilon^{rs},$$

$$I_{12} = a^\alpha_p a^\beta_{qr} a^\gamma_{\beta s} a^\delta_{\alpha\delta} a^\mu_{\gamma\mu}\varepsilon^{pq}\varepsilon^{rs},\ I_{13} = a^\alpha_p a^\beta_{qr} a^\gamma_{\gamma s} a^\delta_{\alpha\beta} a^\mu_{\delta\mu}\varepsilon^{pq}\varepsilon^{rs},$$

$$I_{14} = a^\alpha_p a^\beta_r a^\gamma_{\alpha q} a^\delta_{\beta s} a^\mu_{\gamma\delta} a^\nu_{\mu\nu}\varepsilon^{pq}\varepsilon^{rs},\ I_{15} = a^\alpha_{pr} a^\beta_{qk} a^\gamma_{\alpha s} a^\delta_{\delta l} a^\mu_{\beta\gamma} a^\nu_{\mu\nu}\varepsilon^{pq}\varepsilon^{rs}\varepsilon^{kl},$$

$$I_{16} = a^\alpha_p a^\beta_r a^\gamma_\delta a^\delta_{\alpha q} a^\mu_{\beta s} a^\nu_{\gamma\theta} a^\theta_{\mu\nu}\varepsilon^{pq}\varepsilon^{rs},\ I_{17} = a^\alpha a^\beta_{\alpha\beta},\ I_{18} = a^\alpha a^\beta a^\gamma_\alpha\varepsilon_{\gamma\beta},$$

$$I_{19} = a^\alpha a^\beta_\gamma a^\gamma_{\alpha\beta},\ I_{20} = a^\alpha a^\beta_\alpha a^\gamma_{\beta\gamma},\ I_{21} = a^\alpha a^\beta a^\gamma a^\delta_{\alpha\beta}\varepsilon_{\delta\gamma},$$

$$I_{22} = a^\alpha a^\beta a^\gamma_{\alpha\beta} a^\delta_{\gamma\delta},\ I_{23} = a^\alpha a^\beta a^\gamma_{\alpha\delta} a^\delta_{\beta\gamma},\ I_{24} = a^\alpha a^\beta_\delta a^\gamma_\alpha a^\delta_{\beta\gamma},$$

$$I_{25} = a^\alpha a^\beta_{\beta p} a^\gamma_{\delta q} a^\delta_{\alpha\gamma}\varepsilon^{pq},\ I_{26} = a^\alpha a^\beta_{\beta p} a^\gamma_{\alpha q} a^\delta_{\gamma\delta}\varepsilon^{pq},$$

$$I_{27} = a^\alpha a^\beta a^\gamma a^\delta_\mu a^\mu_{\alpha\beta}\varepsilon_{\delta\gamma},\ I_{28} = a^\alpha a^\beta a^\gamma_\delta a^\delta_{\gamma\mu} a^\mu_{\alpha\beta},$$

$$I_{29} = a^\alpha a^\beta a^\gamma_\mu a^\delta_{\gamma\delta} a^\mu_{\alpha\beta},\ I_{30} = a^\alpha a^\beta_p a^\gamma_{\beta q} a^\delta_{\gamma\mu} a^\mu_{\alpha\delta}\varepsilon^{pq},$$

$$I_{31} = a^\alpha a^\beta_p a^\gamma_{\beta q} a^\delta_{\alpha\gamma} a^\mu_{\delta\mu}\varepsilon^{pq},\ I_{32} = a^\alpha a^\beta_p a^\gamma_{\gamma q} a^\delta_{\alpha\beta} a^\mu_{\delta\mu}\varepsilon^{pq},$$

$$I_{33} = a^\alpha a^\beta a^\gamma a^\delta_{\alpha\beta} a^\mu_{\gamma\nu} a^\nu_{\delta\mu},\ I_{34} = a^\alpha a^\beta a^\gamma_{\alpha p} a^\delta_{\gamma q} a^\mu_{\beta\delta} a^\nu_{\mu\nu}\varepsilon^{pq},$$

$$I_{35} = a^\alpha a^\beta_p a^\gamma_\alpha a^\delta_{\beta q} a^\mu_{\gamma\nu} a^\nu_{\delta\mu}\varepsilon^{pq},\ I_{36} = a^\alpha a^\beta_{pr} a^\gamma_{\alpha q} a^\delta_{\beta s} a^\mu_{\gamma\delta} a^\nu_{\mu\nu}\varepsilon^{pq}\varepsilon^{rs} \tag{1.6}$$

and 33 center-affine comitants, too:

$$K_1 = x^\alpha a^\beta_{\alpha\beta},\ K_2 = x^\alpha x^\beta a^\gamma_\alpha\varepsilon_{\gamma\beta},\ K_3 = x^\alpha a^\beta_\gamma a^\gamma_{\alpha\beta},\ K_4 = x^\alpha a^\beta_\alpha a^\gamma_{\beta\gamma},$$

$$K_5 = x^\alpha x^\beta x^\gamma a^\delta_{\alpha\beta}\varepsilon_{\delta\gamma},\ K_6 = x^\alpha x^\beta a^\gamma_{\alpha\beta} a^\delta_{\gamma\delta},\ K_7 = x^\alpha x^\beta a^\gamma_{\alpha\delta} a^\delta_{\beta\gamma},$$

$$K_8 = x^\alpha a^\beta_\delta a^\gamma_\alpha a^\delta_{\beta\gamma},\ K_9 = x^\alpha a^\beta_{\beta p} a^\gamma_{\delta q} a^\delta_{\alpha\gamma}\varepsilon^{pq},\ K_{10} = x^\alpha a^\beta_{\beta p} a^\gamma_{\alpha q} a^\delta_{\gamma\delta}\varepsilon^{pq},$$

$$K_{11} = x^\alpha x^\beta x^\gamma a^\delta_\mu a^\mu_{\alpha\beta}\varepsilon_{\delta\gamma},\ K_{12} = x^\alpha x^\beta a^\gamma_\delta a^\delta_{\gamma\mu} a^\mu_{\alpha\beta},\ K_{13} = x^\alpha x^\beta a^\gamma_\mu a^\delta_{\gamma\delta} a^\mu_{\alpha\beta},$$

$$K_{14} = x^\alpha a^\beta_p a^\gamma_{\beta q} a^\delta_{\gamma\mu} a^\mu_{\alpha\delta}\varepsilon^{pq},\ K_{15} = x^\alpha a^\beta_p a^\gamma_{\beta q} a^\delta_{\alpha\gamma} a^\mu_{\delta\mu}\varepsilon^{pq},$$

$$K_{16} = x^\alpha a^\beta_p a^\gamma_{\gamma q} a^\delta_{\alpha\beta} a^\mu_{\delta\mu}\varepsilon^{pq},\ K_{17} = x^\alpha x^\beta x^\gamma a^\delta_{\alpha\beta} a^\mu_{\gamma\nu} a^\nu_{\delta\mu},$$

$$K_{18} = x^\alpha x^\beta a^\gamma_{\alpha p} a^\delta_{\gamma q} a^\mu_{\beta\delta} a^\nu_{\mu\nu}\varepsilon^{pq},\ K_{19} = x^\alpha a^\beta_p a^\gamma_\alpha a^\delta_{\beta q} a^\mu_{\gamma\nu} a^\nu_{\delta\mu}\varepsilon^{pq},$$

$$
\begin{aligned}
K_{20} =& x^\alpha a^\beta_{pr} a^\gamma_{\alpha q} a^\delta_{\beta s} a^\mu_{\gamma\delta} a^\nu_{\mu\nu} \varepsilon^{pq}\varepsilon^{rs},\ K_{21} = x^\alpha a^\beta \varepsilon_{\beta\alpha},\ K_{22} = x^\alpha a^\beta a^\gamma_\beta \varepsilon_{\gamma\alpha},\\
K_{23} =& x^\alpha x^\beta a^\gamma a^\delta_{\alpha\beta}\varepsilon_{\gamma\delta},\ K_{24} = x^\alpha x^\beta a^\gamma a^\delta_\mu a^\mu_{\alpha\beta}\varepsilon_{\gamma\delta},\\
K_{25} =& x^\alpha a^\beta a^\gamma a^\delta_{\beta\gamma}\varepsilon_{\delta\alpha},\ K_{26} = x^\alpha a^\beta a^\gamma_{\beta\alpha} a^\delta_{\gamma\delta},\ K_{27} = x^\alpha a^\beta a^\gamma_{\beta\delta} a^\delta_{\gamma\alpha},\\
K_{28} =& x^\alpha a^\beta a^\gamma a^\delta_\mu a^\mu_{\beta\gamma}\varepsilon_{\delta\alpha},\ K_{29} = x^\alpha a^\beta a^\gamma_\mu a^\delta_{\beta\alpha} a^\mu_{\gamma\delta},\ K_{30} = x^\alpha a^\beta a^\gamma_\delta a^\delta_{\beta\alpha} a^\mu_{\gamma\mu},\\
K_{31} =& x^\alpha x^\beta a^\gamma a^\delta_{\gamma\mu} a^\mu_{\nu\delta} a^\nu_{\alpha\beta},\ K_{32} = x^\alpha a^\beta a^\gamma a^\delta_{\beta\gamma} a^\mu_{\nu\alpha} a^\nu_{\mu\delta},\\
K_{33} =& x^\alpha a^\beta a^\gamma_{p\beta} a^\delta_{q\gamma} a^\mu_{\delta\alpha} a^\nu_{\nu\mu}\varepsilon^{pq}, \qquad (1.7)
\end{aligned}
$$

where

$$\varepsilon^{11} = \varepsilon^{22} = \varepsilon_{11} = \varepsilon_{22} = 0, \varepsilon^{12} = -\varepsilon^{21} = \varepsilon_{12} = -\varepsilon_{21} = 1.$$

Using this information let state the following

Proposition 1.3. [1] *Invariants $I_1 - I_{36}$ and comitants $K_1 - K_{33}$ form minimal polynomial basis of center-affine invariants of the quadratic system (1.5).*

Corollary 1.1. *Invariants $I_1 - I_{36}$ form minimal polynomial basis of the center-affine invariants of quadratic system (1.5).*

We remark that the conditions which were expressed through the elements $I_1 - I_{36},\ K_1 - K_{33}$ are not invariants after the shift of the origin of coordinates.

But using qualitative methods for investigating the system (1.5) many authors (see for example [3]) use the affine(linear nonhomogenous) transformation which includes a shift of the origin of coordinates. That is why in order to find corresponding invariant conditions which are expressed through the coefficients of initial system it is neccessary to have affine invariants and comitants.

Let denote by $T(2, R)$ the group of shift transformations of the phase plane of the system (1.5) and let $t \in T(2, R)$,

$$t: \quad x^j = \overline{x}^j + x^j_0 \quad (j = 1, 2).$$

Then the coefficients $\overline{a} = a(t)$ of transformated system will be written as

$$\begin{aligned}
\overline{a}^j &= a^j + a^j_\alpha x^\alpha_0 + a^j_{\alpha\beta} x^\alpha_0 x^\beta_0 \equiv T^j(a, x_0),\\
\overline{a}^j_\alpha &= a^j_\alpha + 2a^j_{\alpha\beta} x^\beta_0 \equiv T^j_\alpha(a, x_0), \qquad (1.8)\\
\overline{a}^j_{\alpha\beta} &= a^j_{\alpha\beta} \equiv T^j_{\alpha\beta}(a, x_0) \quad (j = 1, 2),
\end{aligned}$$

where $x_0 = (x^1_0, x^2_0)$.

Let us write $I_s(a^j,\ a^j_\alpha,\ a^j_{\alpha\beta})$ $(s = 1, 2, \ldots, 36)$ all the elements (1.6) of the writen above minimal polynomial basis of the center-affine polynomial invariants of system (1.5).

Proposition 1.3.[4] *Polynomials*

$$\overline{I}_s(a, x) \equiv I_s(T^j(a, x),\ T^j_\alpha(a, x),\ T^j_{\alpha\beta}(a, x)) \quad (s = 1, 2, ..., 36) \qquad (1.9)$$

form the minimal polynomial basis of the affine comitants of system (1.5).

Thus, any center-affine invariant of any degree can be build as a polynomial from invariants (1.6). By analogy any center-affine comitant can be build as a polynomial from invariants (1.6) and comitants (1.7) and any affine comitant or invariant can be expressed as a polynomial through the affine comitants (1.9).

But it is clear that not every constructed by this method comitant or invariant can be applyied for finding equivalent affine-invariant conditions. For example for invariant I with odd weight the condition $I > 0$ don't have the affine-invariant condition, since any transformation with the negative determinant changes the sign of I.

Let us choose from the set of center-affine comitants those comitants which have the following property: all coefficients from any of these comitants are the absolute invariants of the group $T(2, R)$ of shift transformations. Let us call these comitants T-comitants.

Definition 1.6. *A center-affine comitant $K(a, x)$ is called T- comitant if for every $t \in T(2, R)$ and $a \in R^m$ the following equality holds*

$$K(a(t),\ x) = K(a,\ x).$$

Let us denote by $Aff(2,R)$ the group of affine (linear nonhomogenous) transformations. Since

$$Aff(2,R) = GL(2,R) \bigcup T(2,R)$$

, then it follows that sign-determined T- comitants of even weight, calculated for system (1.5) transformated by any transformation $q \in Aff(2,R)$ remain sign-determined with the same sign. Therefore, T- comitants can be used for obtaining affine-invariant conditions, containing the signs of coefficient expressions.

Clearly, any T-invariant, as a particular case of T-comitant, is an affine invariant of the system (1.5).

Let us consider also "conditional" T-comitants.

Definition 1.7. *A center-affine comitant $K(a,x)$ is called a "conditional" T- comitant if any affine-invariant condition holds.*

It is clear that any T-comitant (as center-affine comitant) can be build as a polynomial from invariants (1.6) and comitants (1.7). From definitions 1 and 6, it follows that any T comitant $K(a,x)$ (or in specific case invariant) which satisfies an affine-invariant condition of type : $K > 0$ or $K < 0$ will essentially satisfy the following conditions:

- have even weight κ;

- have even degree according to coefficients of the system (1.5);

- has the fixed sign form of variables x^1 and x^2.

We remark that the weight κ of any comitant K can be determined using the tensor form which is according to [1, str.18, 28] is determined by

Proposition 1.4. *The weight of a center-affine comitant of the system (1.5) is equal to the number of factors of type ε^{--} with upper indices minus* *the number of factors of type ε_{--} with lower indices which belong to it tensor form.*

Remark 1.1. *By comitant $K(a,x)$ being equal to zero we assume it's identical equality to zero as a polynomial from variables x^1 i x^2, and the sign of the comitant can be determined in any point of the phase plane where comitant $K \neq 0$.*

Condition for determination the number of transversally non-hyperbolic critical points

Let us write the quadratic system (1.5) as

$$\begin{aligned} \frac{dx}{dt} &= a + cx + dy + gx^2 + 2hxy + ky^2 \equiv P(x,y), \\ \frac{dy}{dt} &= b + ex + fy + lx^2 + 2mxy + ny^2 \equiv Q(x,y). \end{aligned} \tag{2.1}$$

From [5] we know that the coordinates (x_0, y_0) of any critical point of system (2.1) are the solution to the system of equations

$$P(x,y) = 0, \quad Q(x,y) = 0. \tag{2.2}$$

Geometrically, this means that any critical point $M_0(x_0, y_0)$ of the system (2.1) is a point of intersection of the curves which can be determined by formulas (2.2). If the parameters of the system are changed and the critical point $M_0(x_0, y_0)$ goes to ∞, then the system (2.1) will have infinitely located critical points which are called Poincare point(see for example [6]) or **transversally non-hiperbolic critical point**. Let us consider the task of obtaining affine invariant conditions for finding the number of those points. Clearly, when the critical point M_0 goes to ∞, the point of intersection of the curves (2.2) goes to ∞. Thus, the branches of these curves intersect at infinity and ,therefore, the curves (2.2) have a common asymptote. From analytical geometry we know that the asymtotes of a second degree curve are determined by its homogeneous quadratic parts.

Thus, our task is to find common linear factors of the homogeneous parts

$$P_2(x,y) = gx^2 + 2hxy + ky^2, \quad Q_2(x,y) = lx^2 + 2mxy + ny^2 \tag{2.3}$$

of the polynomials $P(x,y)$ and $Q(x,y)$ from (2.2) .

Consider the following polynomials defined in terms of invariants invariants (1.6) and comitants (1.7).

$$\mu = \frac{1}{2}(I_8 + I_9 - 2I_7), \quad S = \frac{1}{2}(K_1^2 - K_7), \quad N = \frac{1}{2}(K_1^2 - 2K_6 + K_7) \tag{2.4}$$

and let us prove that the following holds

Theorem 2.1. *The number of transversally non-hiperbolic critical points of system (2.1) is equal to*

0 *if* $\mu \neq 0$;

1 *if* $\mu = 0, \quad S \neq 0$;

2 *(real different) if* $S = 0, N > 0$;

2 *(complex) if* $S = 0, N < 0$;

2 *(real coinciding) if* $N = 0$.

Proof. Calculating the comitants (2.4) for the system (2.1) using (1.6) and (1.7) we find

$$\begin{aligned} \mu =& (gn - kl)^2 - 4(gm - hl)(hn - km) = \\ & (gn - 2hm + kl)^2 - 4(m^2 - ln)(h^2 - gk), \\ S =& (gm - hl)x^2 + (gn - kl)xy + (hn - km)y^2, \\ N =& (m^2 - ln)x^2 + (gn - 2hm + kl)xy + (h^2 - gk)y^2. \end{aligned} \tag{2.5}$$

Constructing the resultant of the homogeneous polynomials $P_2(x,y)$ and $Q_2(x,y)$ ▌ from (2.3) we find that

$$\begin{vmatrix} g & 2h & k & 0 \\ 0 & g & 2h & k \\ l & 2m & n & 0 \\ 0 & l & 2m & n \end{vmatrix} = g \begin{vmatrix} g & 2h & k \\ 2m & n & 0 \\ l & 2m & n \end{vmatrix} - 0 + l \begin{vmatrix} 2h & k & 0 \\ g & 2h & k \\ l & 2m & n \end{vmatrix} - 0 =$$

$$g(-2m\begin{vmatrix} 2h & k \\ 2m & n \end{vmatrix} + n\begin{vmatrix} g & k \\ l & n \end{vmatrix} + 0) + +l(2h\begin{vmatrix} 2h & k \\ 2m & n \end{vmatrix} - k\begin{vmatrix} g & k \\ l & n \end{vmatrix} + 0) =$$

$$g[-2m(2hn-2mk)+n(gn-lk)+0]+l[2h(2hn-2mk)-k(gn-lk)+0]$$

$$= -4gmnh + 4gm^2k + (gn)^2 - gnlk + 4h^2nl - 4mhkl - lkgn - (lk)^2 =$$

$$= (gn)^2 - 2gnlk + (lk)^2 - 4(gmnh - gkm^2 + mhkl - nlh^2)$$

$$= (gn-lk)^2 - 4(gm(nh-km)+hl(km-nh)) = (gn-lk)^2 - 4(gm-hl)(hn-km) = \mu. \blacksquare$$

According to the main theorem about the resultant (see for example [7]) $\mu = 0$ if and only if, when these polynomials have common factor of non-zero degree or $g = l = 0$. In this latter case the common factor is y.

Therefore for $\mu \neq 0$ plane curves (2.2) don't have a common separatrix and the proposition of theorem 2.1 holds.

Suppose condition $\mu = 0$ holds that is $P_2(x,y)$ and $Q_2(x,y)$ have a common (homogeneous) factor of non-zero degree.

1) If $S \neq 0$, then the common factor of the homogeneous polynomials (2.3) can not be of the second degree. Indeed as we can see from (2.5) the coefficients of the comitant S are the minors of the second degree of the matrix coefficients homogeneous of the right sides of the system (2.1). According to $S \neq 0$ at least one comitant is different from zero and thus the order of this matrix is equal to two. This proves that polynomials

$P_2(x,y)$ and $Q_2(x,y)$ are not proportional which means that their common factor is linear which implies that there is

one transversally non-hiperbolic critical point.

2) For $S = 0$, using (2.5) we have

$$gm = hl, \quad gn = kl, \quad hn = km, \tag{2.7}$$

which implies the proportionality of polynomials $P_2(x,y)$ and $Q_2(x,y)$ from (2.3), which states that there exists two constants α and β ($\alpha^2+\beta^2\neq 0$), such that

$$\alpha P_2(x,y)=\beta Q_2(x,y). \tag{2.8}$$

(i)If the following condition holds:

$$k^2+l^2\neq 0$$

then it can be considered that $l\neq 0$ otherwise it can be reached by changing x on y (transfering k in l). Then equality (2.7) gives $h=gm/l,\quad k=gn/l$ and using (2.5) we find

$$N=(m^2-ln)(x-\frac{g}{l}y)^2.$$

Therefore, comitant N becomes sign-determined form and it is clear that condition $N>0(N=0;\ N<0)$ equiavalent to $m^2-ln>0(m^2-ln=0;\ m^2-ln<0)$.

Considering that the expression m^2-ln is a discriminant of a quadratic form $Q_2(x,y)$ (and also a form $P_2(x,y)$ for $\beta\neq 0$ because of (2.8)), we come to a conclusion that if $N>0(N=0;\ N<0)$ then the curves (2.2) have two real different (two real coinciding; two complex) assimptotes".

(ii) If $l=k=0$, then from (2.7) we receive $gm=gn=hn=0$, and so according to (2.5) for comitant N we find

$$N=(mx-hy)^2\geq 0. \tag{2.9}$$

If $N\neq 0$ (then $N>0$), then $m^2+h^2\neq 0$, where we can consider $m\neq 0$ (it can be reached by changing x on y). Using (2.7) we have $l=k=g=hn=0$ and (2.8) can be written as

$$\alpha(2hxy)=\beta(2mxy+ny^2),$$

where $h = \beta = 0$, or $n = 0$. In both cases we get that curves (2.2) have two different real common assymptotes. If $N = 0$, then from (2.9) and (2.7) we have $l = k = m = h = gn = 0$ and then equality (2.8) will be writen as

$$\alpha(gx^2) = \beta(ny^2)$$

and according to $gn = 0$ the common factor of homogeneous polynomials $P_2(x, y)$ and $Q_2(x, y)$ will be x^2 (at $n = 0$), or y^2 (at $g = 0$)

Thus, we get that $S = 0$, $N > 0$($S = N = 0$; $S = 0$, $N < 0$) system (2.1) has two real different (two real coinciding; two complex) transversally non-hiperbolic critical points.

From $S = 0$ it follows that $\mu = 0$,and $N = 0$ implies $\mu = S = 0$. Indeed, as we can see from (2.5), invariant μ is also a discriminant of both comitants S and N, which implies it's equality to zero when we set one of the comitants to zero.

Suppose now that the condition $N = 0$ holds then using (2.5)

$$m^2 - ln = 0, \quad gn - 2hm + kl = 0, \quad h^2 - gk = 0. \qquad (2.10)$$

If $l \neq 0$, then from (2.10) we find that $n = \frac{m^2}{l}$, $k = \frac{gm^2}{l^2}$, $h = \frac{gm}{l}$ and by substitution in (2.5) we get $S = 0$.

Let $l = 0$. Then the first equality (2.10) gives $m = 0$, and the second gives $gn = 0$. If $n = 0$, then from (2.5) it follows that $S = 0$.If $n \neq 0$, then $g = 0$ and the third equality (2.10) gives $h = 0$, and using (2.5) we again come to the equality $S = 0$. **Now theorem 2.1 is proved..** We should say that the conditions of theorem 2.1 are affine-invariant. Indeed, using proposition 1.4 and tensor form of invariants (1.6) and comitants (1.7) we come to a conclusion that invariant μ has weight $\kappa = 2$, and comitant S and N have zero weight. All three center-affine comitants are T-comitants since their coefficients depend only on the homogeneous quadratic parts of the system (2.1) which according to (1.8) remains the same when the center of coordinates is shifted.

The number of ISP and the class of quadratic systems with m_f=1

Consider the system of differential equations:

$$
\begin{aligned}
\frac{dx}{dt} &= a + cx + dy + dx^2 + 2hxy + ky^2 \equiv P(x,y),\\
\frac{dy}{dt} &= b + cx + fy + lx^2 + 2mxy + ny^2 \equiv Q(x,y).
\end{aligned}
\tag{2.1}
$$

Using the notation of [8] and [9] let us denote through m_f the common multiplicity of all singular points (real or complex) located in the finite part of the phase plane of the system (2.1). Since P(x,y) and Q(x,y) are the curves of second degree it follows that

$$0 \leq m_f \leq 4.$$

We denote by $R_{jl} resp. (C_{jl})$ any real resp. (complex) infinite singular point of multiplicity $j+l$, which bifurcates into j finite singular points and l infinite singular points. Let d_{rs} denote the second order minor of the matrix made up of the coefficients of the right-hand sides of the system (2.1), which are formed by the columns r and s:

$$
\begin{aligned}
&d_{12} = ae - bc, \quad d_{13} = af - bd, \quad d_{14} = al - bg,\\
&d_{15} = am - bh, \quad d_{16} = an - bk, \quad d_{23} = cf - de,\\
&d_{24} = cl - eg, \quad d_{25} = cm - eh, \quad d_{26} = cn - ek,\\
&d_{34} = dl - fg, \quad d_{35} = dm - fh, \quad d_{36} = dn - fk,\\
&d_{45} = gm - hl, \quad d_{46} = gn - hk, \quad d_{56} = hn - km.
\end{aligned}
\tag{3.1}
$$

In [8] the affine-invariant coefficient conditions determining the number m_f have been found and the following comitants and invariants of the system

(2.1) were used:

$$
\begin{aligned}
2\mu = {} & I_8 + I_9 - 2I_7,\\
2H = {} & -I_1K_9 - I_4K_1 + I_5K_1 - 2K_{14} + 2K15,\\
4G = {} & 2I_1^2K_6 + 4I_1^2K_7 - 4I_1K_1K_3 - 4I_1K_1K_4 - 2I_1K_{12} - 2I_1K_{13} + I_2K_1^2\\
& + 2I_2K_6 - 5I_2K_7 - 4I_3K_2 + 6I_4K_2 + 2I_5K_2 + 8I_{17}K_1^2 - 8I_{17}K_6 + 8I_{17}K_7\\
& - 16K_1K_{27} + 4K_3K_3 + 4K_4^2 + 8K31,\\
2F = {} & -I_1^3K_5 + I_1^2K_{11} + I_1I_2K_5 - 4I_1K_1K_{23} - I_2K_{11} - 8I_{17}K_1K_2 + 8I_{17}K_{11}\\
& - 8I_{19}K_5 + 4K_1^2K_{22} + 8K_1K_{24} + 8K_2K_{27} - 4K_3K_{23} - 4K_7K_{22},\\
V = {} & I_1^2K_5K_{21} - I_1K_2K_{23} - 2I_1K_5K_{22} + K_{11}K_{22} + K_{23}^2. \qquad (3.2)
\end{aligned}
$$

Using (1.6), (1.7) and (3.1) it is easy to find comitant expressions for (3.2) in terms of the coefficients of the system (2.1):

$$
\begin{aligned}
\mu = {} & d_{46}^2 - 4d_{45}d_{56},\\
H = {} & (2d_{45}d_{35} - 2d_{45}d_{26} + d_{46}d_{25} - d_{46}d_{34})x +\\
& (d_{46}d_{35} - d_{46}d_{26} + 2d_{56}d_{25} - 2d_{56}d_{34})y,\\
G = {} & (d_{24}d_{26} - d_{24}d_{35} + d_{34}^2 - d_{34}d_{25} - 3d_{45}d_{23} + 4d_{45}d_{15} - 2d_{46}d_{14})x^2 +\\
& (2d_{25}d_{26} - 4d_{25}d_{35} + 2d_{34}d_{35} - 3d_{23}d_{46} + 4d_{45}d_{16} - 4d_{56}d_{14})xy +\\
& (d_{26}^2 - d_{35}d_{26} + d_{34}d_{36} - d_{25}d_{36} - 3d_{23}d_{56} + 2d_{46}d_{16} - 4d_{56}d_{15})y^2,\\
F = {} & (d_{23}d_{24} + 4d_{12}d_{45} + 2d_{14}d_{34} - 2d_{14}d_{25})x^3 + (d_{23}d_{34} + 2d_{23}d_{25} +\\
& 4d_{13}d_{45} + 4d_{12}d_{46} + 2d_{14}d_{35} - 2d_{14}d_{26} + 4d_{15}d_{34} - 4d_{15}d_{25})x^2y +\\
& (d_{23}d_{26} + 2d_{23}d_{35} + 4d_{13}d_{46} + 4d_{12}d_{56} + 4d_{15}d_{35} - 4d_{15}d_{26} +\\
& 2d_{16}d_{34} - 2d_{16}d_{25})xy^2 + (d_{23}d_{36} + 4d_{13}d_{56} + 2d_{16}d_{35} - 2d_{16}d_{26})y^3,\\
V = {} & (d_{14}^2 - d_{12}d_{24})x^4 + (4d_{14}d_{15} - d_{12}d_{34} - 2d_{12}d_{25} - d_{13}d_{24})x^3y +\\
& (2d_{14}d_{16} + 4d_{15}^2 - d_{12}d_{26} - 2d_{12}d_{35} - d_{13}d_{34} - 2d_{13}d_{25})x^2y^2 +\\
& (4d_{15}d_{16} - d_{12}d_{36} - d_{13}d_{26} - 2d_{13}d_{35})xy^3 + (d_{16}^2 - d_{13}d_{36})y^4 \qquad (3.4)
\end{aligned}
$$

According to [8], let us use the following theorem:

Theorem 3.1. *The number m_f, which is equal to the common multiplicity of all critical points located in the finite part of the phase plane of the quadratic system (2.1) has values:*

$$
\begin{aligned}
&m_f = 4 \quad \text{iff} \quad \mu \neq 0;\\
&m_f = 3 \quad \text{iff} \quad \mu = 0, \quad H \neq 0;\\
&m_f = 2 \quad \text{iff} \quad \mu = H = 0, \quad G \neq 0;\\
&m_f = 1 \quad \text{iff} \quad \mu = H = G = 0, \quad F \neq 0;\\
&m_f = 0 \quad \text{iff} \quad \mu = H = G = F = 0, \quad V \neq 0;\\
&m_f = \infty \quad \text{iff} \quad \mu = H = G = F = V = 0.
\end{aligned}
$$

Definition 3.1. *The system of differential equations is called degenerated if its right sides have the common factor of nonzero degree.*

From theorem 3.1 we obtain the following:

Corollary 3.1. *The quadratic system of differential equations (2.1) is degenerated, if and only if, the following conditions are satisfied:*

$$\mu = H = G = F = V = 0.$$

Let us introduce the following comitant of the quadratic system

$$L = K_5, \quad 2M = K_1^2 + 6K_6 - 9K_7, \quad \eta = 18I_7 - 27I_8 + I_9 \tag{3.5}$$

and consider the infinite singular points. As we know (see [1]) the infinitely located critical points of the quadratic system (2.1) are located on the "extremeties" (or "ends") of the integral lines of the corresponding homogenous quadratic system

$$
\begin{aligned}
\frac{dx}{dt} &= gx^2 + 2hxy + ky^2 \equiv P_2(x,y),\\
\frac{dy}{dt} &= lx^2 + 2mxy + ny^2 \equiv Q_2(x,y),
\end{aligned}
\tag{3.6}
$$

These integral lines are defined by equation

$$L \equiv yP_2(x,y) - xQ_2(x,y) = -lx^3 + (g-2m)x^2y + (2h-n)xy^2 + ky^3 = 0. \quad (3.7)$$

In [1] necessary and sufficient center affine conditions for determining the number of infinite critical points were found. However we must remark that there are eight series of such conditions in this work. Let us find more natural equivalent necessary and sufficient center affine conditions which are different than the ones from [1].

We have

Lemma 3.1. *The number of real infinitely located critical points of a non-degenerate quadratic system (2.1) is equal to*

$$\begin{array}{ll} 3 & \textit{iff} \ \ \eta > 0; \\ 2 & \textit{iff} \ \ \eta = 0, \ M \neq 0; \\ 1 & \textit{iff} \ \ \eta < 0 or \quad M = 0, \ L \neq 0; \\ \infty & \textit{iff} \ \ L = 0. \end{array}$$

Proof. It is easy to check that the comitant $4M$ is actually a Hessian of the form L, i.e.

$$M = \frac{1}{4}\begin{vmatrix} \frac{\partial^2 L}{\partial x^2} & \frac{\partial^2 L}{\partial x \partial y} \\ \frac{\partial^2 L}{\partial y \partial x} & \frac{\partial^2 L}{\partial y^2} \end{vmatrix} =$$

$$= \frac{1}{4}[(-6lx + 2(g-2m)y)(2(2h-n)x + 6ky) - (2(g-2m)x + 2(2h-n)y)^2] =$$

$$= [3l(n-2h) - (g-2m)^2]x^2 + [(g-2m)(n-2h) - 9kl]xy$$
$$+ [3k(g-2m) - (n-2h)^2]y^2 \quad (3.8)$$

and the invariant η is the discriminant of the same form:

$$\eta = (g-2m)^2(2h-n)^2 + 4l(2h-n)^3 - 27k^2l^2 - 4k(g-2m)^3$$
$$- 18kl(g-2m)(2h-n). \quad (3.9)$$

We should remark that if the condition $L = 0$ is satisfied, then the homogeneous system (3.6) has an infinite number of integral lines [1]. Thus, when $L = 0$ the system (2.1) has an infinite number of critical points (these points are located on the "ends" of the integral lines).

Let $L \neq 0$. We have two cases:

1) If the discriminant of the form L satisfies $\eta > 0$ ($\eta < 0$) the form L has three (one) real linear homogeneous factors and therefore the system (2.1) has three (one) real infinite points.

2) Let $\eta = 0$. In this case the form L has a linear real factor of multiplicity greater than one. However, (cf [7]), if the Hessian of a two-dimensional form F of degree m is equal to zero, then such form may be writen as

$$F = (\alpha x + \beta y)^m,$$

where α and β are real numbers. Hence, if Hessian M is of the form $M = 0$ then

$$L = (\alpha x + \beta y)^3$$

and system (2.1) has only one infinite singular point (of the multiplicity 3). Otherwise (i.e. $M \neq 0$) the form L with vanishing discriminant η may be expressed in the form

$$L = (\alpha x + \beta y)^2 (\gamma x + \delta y),$$

where α, β, γ and δ are real numbers. Therefore system (2.1) has two different real infinite singular points (one of which is of multiplicity two). Lemma 3.1 is proved.

Determining the character of critical point of quadratic system

As we know from [5], in the investigation of the character of the trajectories of the system (2.1) on its phase plane, the determination of the characters of the critical points plays the most important role. The problem is, how to determine the trajectories behaviour in a small neibourhood of each critical point.

Let $M(x_0, y_0)$ be a critical point of the system (2.1), so we have

$$P(x_0, y_0) = Q(x_0, y_0) = 0.$$

Let us use the notation from [5], and set

$$\Delta(x_0, y_0) = \begin{vmatrix} P'_x(x_0, y_0) & P'_y(x_0, y_0) \\ Q'_x(x_0, y_0) & Q'_y(x_0, y_0) \end{vmatrix}$$

$$\sigma = P'_x(x_0, y_0) + Q'_y(x_0, y_0) \tag{4.1}$$

According to [5] the condition of equilibrium $M(x_0, y_0)$ is called simple, if it satisfies the following relation:

$$\Delta(x_0, y_0) \neq 0.$$

Applying lemma from [5]

Lemma 4.1. *The point* $M(x_0, y_0)$ *is a*

– *saddle if* $\Delta < 0$;

– *node if* $\Delta > 0, \quad \sigma^2 - 4\Delta \geq 0$;

– *focus if* $\Delta > 0, \quad \sigma^2 - 4\Delta < 0, \quad \sigma \neq 0$;

– *focus or center if* $\Delta > 0, \quad \sigma^2 - 4\Delta < 0, \quad \sigma = 0$.

We should remark that the problem of distinguishing the center and focus by coefficient conditions is a classical problem of tremendous difficulty. At the present time it is solved fully only when polynomials $P(x, y)$ and $Q(x, y)$ are of the second degree (cf.[10]) , and also for some specific cases for polynomials of the third degree.

Now let us consider the determination of the character of the critical point of a quadratic system with $m_f = 1$ and two real coinciding transversally non-hiperbolic critical points. The first step in this direction is to find the corresponding canonical form of the system (2.1).

According to theorem 2.1 the system (2.1) has two real coinciding transversally non-hiperbolic critical points if and only if the condition $N = 0$ holds. In this case, as it has been shown in the proof of theorem 2.1, the homogeneous polynomials $P_2(x,y)$ and $Q_2(x,y)$ have common twofold linear factors of non-zero degree, i.e.

$$P_2(x,y) = \alpha(px+qy)^2, \quad Q_2(x,y) = \beta(px+qy)^2.$$

In order to have at least one summation term, it is neccessary that the condition $p^2 + q^2 \neq 0$ be satisfied. At this point, it is always possible to consider that $p = 0,\ q \neq 0$, by applying to the system (2.1) the transformation

$$x_1 = x, \quad y_1 = px + qy$$

if $q \neq 0,\ p \neq 0$ or by exchange x and y if $q = 0$ (then $p \neq 0$).

In both cases the system (2.1) will be written as

$$\frac{dx}{dt} = a + cx + dy + ky^2, \quad \frac{dy}{dt} = b + ex + fy + ny^2, \tag{4.2}$$

Applying (3.1) we find

$$d_{14} = d_{24} = d_{34} = d_{15} = d_{25} = d_{35} = d_{45} = d_{46} = d_{56} = 0.$$

Using (3.2) and (3.4) we get

$$\begin{aligned} \mu =& 0,\ H = 0,\ G = d_{26}^2 y^2,\ F = d_{23}d_{26}xy^2 + (d_{23}d_{36} - 2d_{16}d_{26})y^3, \\ \eta =& 0,\ M = -n^2y^2,\ L = (-nx + ky)y^2. \end{aligned} \tag{4.3}$$

According to theorem 3.1, for the system (2.1) the condition $m_f = 1$ holds if and only if $\mu = H = G = 0,\ F \neq 0$. Therefore from (4.3) and $G = 0$ it follows that $d_{26} = 0$, and then $F = d_{23}d_{36}y^3 \neq 0$. And thus, for the system (4.2) with $m_f = 1$ the following conditions must hold:

$$d_{26} = cn - ek = 0,\ F = d_{23}d_{36}y^3 = (cf - de)(dn - fk)y^3 \neq 0. \tag{4.4}$$

The coordinates of the single critical point, located in the finite phase plane of system (4.2), will satisfy the system of equations:

$$a + cx + dy + ky^2 = 0, \quad b + ex + fy + ny^2 = 0. \tag{4.5}$$

Multiplying the first equation (4.5) by n, the second one by $-k$ and adding these equations we have

$$d_{16} + d_{26}x + d_{36}y = 0,$$

which gives $y_0 = -d_{16}/d_{36}$. (Note that (4.4) implies $d_{36} \neq 0$.)

By analogy multiplying the first equation (4.5) by f, the second one by $-d$ and adding these equations we come up with an equation which determines the axis of a stationary point:

$$d_{13} + d_{23}x - d_{36}y_0^2 = 0.$$

Thus, using the value for y_0 we come up to the following coordinates of the stationary point $M(x_0, y_0)$ of system (4.2):

$$x_0 = \frac{d_{16}^2 - d_{13}d_{36}}{d_{23}d_{36}}, \qquad y_0 = -\frac{d_{16}}{d_{36}}. \tag{4.6}$$

Then, for the system (4.2), (4.6) implies that

$$\begin{aligned} P'_x(x_0, y_0) &= c, \quad P'_y(x_0, y_0) = d + 2ky_0 = \frac{1}{d_{36}}(d_{36}d - 2d_{16}k), \\ Q'_x(x_0, y_0) &= e, \quad Q'_y(x_0, y_0) = f + 2ny_0 = \frac{1}{d_{36}}(d_{36}f - 2d_{16}n) \end{aligned}$$

Applying (4.1) and (4.4) we find that

$$\begin{aligned} \Delta &= \frac{c}{d_{36}}(d_{36}f - 2d_{16}n) - \frac{e}{d_{36}}(d_{36}d - 2d_{16}k) \\ &= \frac{1}{d_{36}}[d_{36}(cf - de) - 2d_{16}(cn - ek)] = cf - de = d_{23}, \\ \sigma &= \frac{1}{d_{36}}[d_{36}(c + f) - 2d_{16}n]. \end{aligned} \tag{4.7}$$

In order to make our next steps simpler consider two cases: Comitant $M \neq 0$ and $M = 0$.

Case 1. Suppose the condition $M \neq 0$ holds. Applying (4.3) we have $n \neq 0$. Hence, for the system (4.2) we can assume that the conditon $k = 0$ holds, by applying the linear transformation $x_1 = nx - ky$, $y_1 = y$. From (4.4) using $n \neq 0$ we find $c = 0$, and so the system (4.2) can be written as

$$\frac{dx}{dt} = a + dy, \quad \frac{dy}{dt} = b + ex + fy + ny^2, \tag{4.8}$$

From (4.4) and (4.7) we find that

$$F = -d^2eny^3 \neq 0, \quad \Delta = -de, \quad \sigma = \frac{1}{d}(f - 2an). \tag{4.9}$$

Calculating the values of the expression $\sigma^2 - 4\Delta$ for the system (4.8) and taking into account (4.9) we find

$$\sigma^2 - 4\Delta = \frac{1}{d^2}[(df - 2an)^2 + 4d^3e]. \tag{4.10}$$

Let us consider the center-affine invariants

$$G_1 = (I_1^2 - I_2), \quad G_2 = I_1I_2 - I_1^3 + 4I_1I_{17} - 4I_{20}, \tag{4.11}$$

which, for system (4.8), have the following values:

$$G_1 = -2de, \quad G_2 = 2e(df - 2an). \tag{4.12}$$

Let us show that applying to the system (2.1) the conditions $N = G = M = 0$, forces the invariants G_1 and G_2 to become condition T- comitants , i.e. invariants when the center of coordinates is shifted. Indeed, as it was shown above, when these conditions hold the system (2.1) can be written as (4.8). Now it is left to show that expressions (4.12) are invariant when the center of coordinates of the system (4.8) phase plane is shifted. Substituting $x =$

$x_1 + x_0,\ y = y_1 + y_0$ for a point (x_0, y_0) of the phase plane of system (4.8) we obtain

$$\frac{dx_1}{dt} = a_1 + d_1 y_1, \quad \frac{dy_1}{dt} = b_1 + e_1 x_1 + f_1 y_1 + n_1 y_1^2$$

with

$$a_1 = a + dy_0,\ d_1 = d,\ b_1 = b + ex_0 + fy_0 + ny_0^2,\ e_1 = e, f_1 = f + 2ny_0,\ n_1 = n.$$

Thus, calculating the values for the invariants G_1 and G_2 we find

$$G_1 = -2d_1e_1 = -2de, \quad G_2 = 2e_1(d_1 f_1 - 2a_1 n_1)$$

$$= 2e[d(f + 2ny_0) - 2(a + dy_0)n] = 2e(df - 2an),$$

which proves that G_1, G_2 are condition T-comitants. However we must remark that, in general, invariants G_1 and G_2 are not T-comitants for the system (2.1).

Let us consider $G_2^2 - 2G_1^3$ and calculate its value for the system (4.8). Using (4.12) we find

$$G_2^2 - 2G_1^3 = 4e^2[(df - 2an)^2 + 4d^3e]. \tag{4.13}$$

Now it is easy to prove the following theorem:

Theorem 4.1. *The single equilibrium state of a system (2.1) with $M \neq 0$, $m_f = 1$ and two coinciding transversally non-hiperbolic critical points is a*

– *saddle if* $G_1 < 0$;

– *node if* $G_1 > 0, \quad G_2^2 - 2G_1^3 \geq 0$;

– *focus if* $G_1 > 0, \quad G_2^2 - 2G_1^3 < 0, \quad G_2 \neq 0$;

– *center if* $G_1 > 0, \quad G_2^2 - 2G_1^3 < 0, \quad G_2 = 0$.

By virtue of proposition 1.4, (1.6) and (4.11) the invariants G_1 and $G_2^2 - 2G_1^3$ have zero weight and even degree with respect to the coefficients of the system (2.1). Applying (4.9), (4.10), (4.12) and (4.13) we find that

$$SgnG_1 = Sgn\Delta, \quad Sgn(G_2^2 - 2G_1^3) = Sgn(\sigma^2 - 4\Delta)$$

Hence, the condition $G_2 = 0$ is equivalent to $\sigma = 0$. Thus, using lemma 4.1 we conclude that the critical point is a saddle (resp. node; focus) when

$$G_1 < 0 \ (resp. G_1 > 0,\ G_2^2 - 2G_1^3 > 0;\ G_1 > 0,\ G_2^2 - 2G_1^3 < 0,\ G_2 \neq 0).$$

In order to complete the proof of the theorem 4.1 it is left to show that when the following condition holds

$$G_1 > 0, \quad G_2^2 - 2G_1^3 < 0, \quad G_2 = 0 \tag{4.14}$$

the critical point of (4.8) is a center. If the conditions (4.14) hold, which are affine invariant (as it was shown above), then applying $k = c = 0$ from (4.6) we get the coordinate of a critical point of a system (4.8):

$$x_0 = \frac{a(fd - an) - bd^2}{d^2 e}, \quad y_0 = -\frac{a}{d}.$$

Applying the shift transformation $x_1 = x - x_0$, $y_1 = y - y_0$, the center of a coordinate system moves to a point (x_0, y_0) which leads the system (4.8) to a system

$$\frac{dx}{dt} = dy, \quad \frac{dy}{dt} = ex + (f - 2\frac{an}{d})y + ny^2, \tag{4.15}$$

for which (1.6) gives

$$I_1 = \frac{df - 2an}{d}, \ I_2 = \frac{1}{d^2}[(df - 2an)^2 + 2d^3 e], \ I_6 = \frac{n^2 e}{d}(df - 2an), \ I_{13} = 0.$$

Then the condition $G_2 = 2e(df - 2an) = 0$ gives that $I_1 = I_6 = 0, I_2 = 2de$. Since $G_2^2 - 2G_1^3 = 16d^3e^3$, according to (4.14) we have $de < 0$, i.e. $I_2 < 0$. Hence, the condition $I_1 = I_6 = I_{13} = 0$, $I_2 < 0$ holds for the system (4.15) and according to theorem 1.34 from [1, p.131] the critical point O(0,0) of this system is a center . Thus, theorem 4.1 is proved.

Case 2. Now assume the condition $M = 0$ hold. Using (4.3) we have $n = 0$ and condition $cn - ek = 0$, $F \neq 0$ according to (4.6) implies that

$k \neq 0,\ e = 0$. Thus, the system (4.2) can be written in the form

$$\frac{dx}{dt} = a + cx + dy + ky^2, \quad \frac{dy}{dt} = b + fy, \tag{4.16}$$

for which

$$\begin{aligned} F =& - cf^2ky^3 \neq 0, \quad \Delta = cf, \quad \sigma = c + f, \quad \sigma^2 - 4\Delta = (c - f)^2, \\ G_1 =& 2cf, \quad G_2 = -2cf(c + f), \quad G_2^2 - 2G_1^3 = 4c^2f^2(c - f)^2. \end{aligned} \tag{4.17}$$

Since $\sigma^2 - 4\Delta = (c - f)^2 \geq 0$, then lemma 4.1 and (4.17) imply that the following theorem holds.

Theorem 4.2. *Single singular point of a system (2.1) with $M = 0$, $m_f = 1$ and two coinciding transversally non-hiperbolic critical points is a saddle when $G_1 < 0$, and a node when $G_1 > 0$.*

Condition of topological classification of quadratic system with $m_f = 1$ and two coinciding transversally non-hyperbolic critical points

We know from [5], that the system (2.1) defines, in general, some set of trajectories in the xy-plane

with a well-defined direction of motion (to the appropriate increase in time t).

We shall consider the hemisphere S

$$x^2 + y^2 + (z - 1)^2 = 1, \quad z \leq 1. \tag{5.1}$$

Let us define a central projection of the xy-plane onto S with centre of the projection at the centre of the hemisphere $O_1(0, 0, 1)$, that is we join the point O_1 with arbitrary point $M \in$ xy-plane. The closed interval $\overline{O_1M}$ cuts the hemisphere S at a unique point $M_1 \in S$, which we then place in correspondence with the point M. Then a subset of the xy-plane will correspond to the same subset of the hemisphere (5.1) with $z < 1$, while the circumference $z = 1$ (the equator) will correspond to points at infinity on the plane. If we project, finally, the hemisphere (5.1) by an orthogonal projection onto the xy-plane

we obtain the circle K from some set of curves with direction given to each of them by the so-called Poincare disc of system (2.1).

Definition 5.1. *We shall say that two systems of the form (2.1) are* **topologically equivalent**, *if there exists a homeomorphism (one to one, on both sides continuous transformation) of their Poincare discs, transforming the loci of one system into the loci of the other.*

The topological classification of the quadratic system (2.1) with a unique real critical point on its phase plane is considered in [11] . The conditions for the classification of the topological classes obtained in [11] are expressed through the parameters of the corresponding canonical forms. Because of that we can not use these conditions for determining the topological class of the initial system. In [12] topological quadratic system is given for which the condition $m_f = 1$ holds, i.e. three critical points (including imaginary) went from the finite part of the plane to infinitely located points. But not all the known conditions for distinguishing topological classes are expressed directly through the coefficients of the initial system and because of that our task of applying them becomes more difficult. The conditions which were found in [12] are not invariant when affine transformations are applied. Below will be found affine invariant coefficient condition for the topological classification of a quadratic systems such as

$$\begin{aligned} \frac{dx}{dt} &= a + cx + dy + gx^2 + 2hxy + ky^2, \\ \frac{dy}{dt} &= b + ex + fy + lx^2 + 2mxy + ny^2, \end{aligned} \tag{5.1}$$

with $m_f = 1$, and having two coinciding transversally non-hiperbolic critical points. According to paragraph 2 and 3 the system (5.1) is a system with

$m_f = 1$, having coinciding transversally non-hiperbolic critical points, if and only if, the following affine invariant conditions hold:

$$\mu = H = G = N = 0, \quad F \neq 0. \tag{5.2}$$

Let us consider the following polynomial on the comitants (1.7) and invariants (1.6):

$$W = 3I_2K_5^2 - I_1K_5K_{11} - 3K_{11}^2. \tag{5.3}$$

Then the following theorem holds:

Theorem 5.1. *The qualitative illustration of quadratic system (5.1) with conditions (5.2) in the Poincare disc with accuracy of topological equivalency are settled by:*

– *picture 1 when* $M \neq 0, \quad G_1 < 0;$

– *picture 2 when* $M \neq 0, \quad G_1 > 0, \quad G_2 \neq 0;$

– *picture 3 when* $M \neq 0, \quad G_1 > 0, \quad G_2 = 0;$

– *picture 4 when* $M = 0, \quad G_1 < 0;$

– *picture 5 when* $M = 0, \quad G_1 > 0, \quad W \geq 0;$

– *picture 6 when* $M = 0, \quad G_1 > 0, \quad W < 0.$

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picture. 1 - Figure 3a;

picture. 2 - Figure 3b ($\beta \neq 0$);

picture. 3 - Figure 3b ($\beta = 0$);

picture. 4 - Figure 3c;

picture. 5 - Figure 3d (a);

picture. 6 - Figure 3d (b);

Proof. As it was proven in paragraph 4 when the condition (5.2) holds the system (5.1) can be written as (4.2) with conditions (4.4).

1) If $M \neq 0$, then there exists an affine transformation (see p. 4), which changes the system (5.1) into (4.8), i.e. into the system:

$$\frac{dx}{dt} = a + dy, \quad \frac{dy}{dt} = b + ex + fy + ny^2,$$

for which the point $(x_0,\ y_0)$ with the coordinates

$$x_0 = \frac{a(fd - an) - bd^2}{d^2 e}, \quad y_0 = -\frac{a}{d}$$

is a unique critical point. After shifting the center of coordinates in this point we obtain the system:

$$\frac{dx}{dt} = dy, \quad \frac{dy}{dt} = ex + fy + ny^2, \tag{5.4}$$

for which

$$F = -d^2 eny^3 \neq 0, \quad G_1 = -2de. \tag{5.5}$$

Applying the transformation

$$x_1 = \frac{n}{d} sgn(de) x, \quad y_1 = \frac{n}{\sqrt{|de|}} y, \quad \tau = \sqrt{|de|} t,$$

and considering the equality $sgn(de) = -sgnG_1$, the system (5.4) can be written as

$$\frac{dx_1}{d\tau} = -y_1 sgnG_1, \quad \frac{dy_1}{d\tau} = x_1 + \beta y_1 + y_1^2. \tag{5.6}$$

a) If $G_1 < 0$, then according to Theorem 4.1 the point $(0,0)$ of a system (5.6) is called a *saddle* and our system is going to be as the system (16) from [12], in which it was proved that phase portrait of this system corresponds to picture 1

(Figure 3a from [12]).

b) If $G_1 > 0$, then the system (5.6) becomes as system (17) from [12], i.e. we obtain the system:

$$\frac{dx}{dt} = -y, \quad \frac{dy}{dt} = x + \beta y + y^2, \tag{5.7}$$

for which

$$G_1 = 2, \quad G_2 == -2\beta, \quad G_2^2 - 2G_1^3 = 4(\beta^2 - 4). \tag{5.8}$$

According to Theorem 4.1 the critical point $(0,0)$ of the system (5.8) is called a *node* when $G_2^2 - 2G_1^3 \geq 0$, a *focus* when $G_2^2 - 2G_1^3 < 0$, $G_2 \neq 0$ and a *center* when $G_2^2 - 2G_1^3 < 0$, $G_2 = 0$. Since a local neighborhood of a critical point of focus type is equivivalent to neighborhood of a critical point of node type (see [5]) then according to [12] the qualitative illustration of the system (5.7) corresponds to the picture 2 (Figure 3b $(\beta \neq 0)$) from [12]) when $\beta \neq 0$ and to picture 3 (Figire 3b $(\beta = 0)$) when $\beta = 0$. Finally, notice, that using (5.8) condition $\beta \neq 0$ $(\beta = 0)$ is equivalent to $G_2 \neq 0$ $(G_2 = 0)$.

2) Let $M = 0$ holds. Then as it was shown in p. 4, the system (5.1) leads to the system (4.16), which has the critical points (x_0, y_0) with the coordinates

$$x_0 = \frac{b(df - bk) - af^2}{cf^2}, \quad y_0 = -\frac{b}{f}.$$

After shifting the origin to this point the system (4.16) leads to the system

$$\frac{dx}{dt} = cx + dy + ky^2, \quad \frac{dy}{dt} = fy, \tag{5.9}$$

for which

$$F = -cf^2ky^3 \neq 0, \quad G_1 = 2cf. \tag{5.10}$$

Then the transformation

$$x_1 = \frac{f}{k}x, \quad y_1 = y, \quad d\tau = fdt$$

appled to the system (5.9) leads to

$$\frac{dx_1}{d\tau} = cx_1 + \beta y_1 + y_1^2, \quad \frac{dy_1}{d\tau} = y_1, \tag{5.11}$$

for which we have $G_1 = 2c \neq 0$. According to theorem 4.2 critical point (0,0) of the system (5.11) is called a saddle when $G_1 < 0$ $(c < 0)$ and node when

$G_1 > 0$ $(c > 0)$. In fact it is easy to see, that system (5.11) is the same as the system (18) from [12] when $c = \bar{a}_{10}/\bar{b}_{01}$. According [12] for the system (5.11) there are only tree qualitative illustrations: picture 4 (Figure 3c) when $\bar{a}_{10}/\bar{b}_{01} < 0$, picture. 5 (Figure 3d (a)) when

$$\bar{a}_{10}/\bar{b}_{01} > 0, \quad \bar{b}_{01}(2\bar{b}_{01} - \bar{a}_{10}) \geq 0$$

and picture 6 (Figure 3d (b)) when

$$\bar{a}_{10}/\bar{b}_{01} > 0, \quad \bar{b}_{01}(2\bar{b}_{01} - \bar{a}_{10}) < 0.$$

Since $G_1 = 2c = 2\bar{a}_{10}/\bar{b}_{01}$, then it is obvious, that the condition $\bar{a}_{10}/\bar{b}_{01} > 0$ $(\bar{a}_{10}/\bar{b}_{01} < 0)$ is equivalent to $G_1 > 0$ $(G_1 < 0)$. We need to express in terms of the comitants and invariants of the quadratic system the condition

$$\bar{b}_{01}(2\bar{b}_{01} - \bar{a}_{10}) \geq 0, \ (\bar{b}_{01}(2\bar{b}_{01} - \bar{a}_{10}) < 0). \tag{5.12}$$

Let us expressed (5.12) throgh the parameters of the system (5.11). Since $c = \bar{a}_{10}/\bar{b}_{01}$, we have

$$\bar{b}_{01}(2\bar{b}_{01} - \bar{a}_{10}) = \bar{b}_{01}^2(2 - \frac{\bar{a}_{10}}{\bar{b}_{01}}) = \bar{b}_{01}^2(2 - c).$$

Thus, the condition (5.12) is equivalent to $2 - c \geq 0$ $(2 - c < 0)$.

Calculating for system (5.11) invariants (1.6) and comitants (1.7) we conclude that the only non-zero elements will be

$$I_1 = c + 1, \ I_2 = c^2 + 1, \ K_2 = (c-1)x_1y_1 + \beta y_1^2, \ K_5 = y_1^3, K_{11} = cy_1^3, \tag{5.13}$$

all others are equal to zero. Since I_1 is of the first degree, according to coefficients of initial system (5.1), and I_2 is of the second degree, then any polynomial on the invariants (5.13) having even degree on the coefficients of the system and having sign $2 - c$, is called polynomial with two variables: $I_1^2 = (c+1)^2$

and $I_2 = c^2 + 1$. That's why it is easy to see that it is impossible to obtain polynomial, containing the factor

$2-c$ or $c(2-c)$ (since the sign of the parameter c is determined by invariant G_1) by such method.

Thus, we conclude that it is neccessary to use the comitants K_5 and K_{11} (K_2depends on parameter β, which doesn't belonf to

our condition).Also, we have to consider the requirements to the comitants, mentioned in paragraph one applied in conditions of type $K \geq 0$, i.e.: having even weight, having even degree on the coefficients of the system and having fixed sign form on the phase variables (in our case

x_1 and y_1).

According to (1.7) and proposition 1.4 the comitant K_5 $(resp.K_{11})$ has weight $\kappa = -1(resp. -1)$, and degree equal to 1 (resp. 2) on the coefficients of the initial system and degree 3 (resp. 3) on the phase varaibles. Using (5.13) and considering the expressions

$$\begin{aligned} &I_1^2K_5^2 = (c+1)^2y_1^6, \quad I_2K_5^2 = (c^2+1)y_1^6, \\ &I_1K_5K_{11} = (c+1)y_1^6, \quad K_{11}^2 = c^2y_1^6, \end{aligned} \tag{5.14}$$

we obtain four fixed sign forms (since they have even (sixth) degree according to x_1 and y_1 and contain only y_1^6). Each of them has the even weight $\kappa = -2$ and even degree on the coefficients of the initial system. Using the method of un-determined coefficients we find a linear combination

of expressions (5.14), which will be equal to $(2-c)y_1^6$. It is easy to see that the expression

$$W = 3I_2K_5^2 - I_1K_5K_{11} - 3K_{11}^2 = (2-c)y_1^6.$$

is exactly the expression we were looking for. Thus, the invariant condition is found. Notice, that the condition $W \geq 0$ $(W < 0)$ is affine invariant, i.e. it is invariant when the origin is shifted. And this is obvious since the system (5.11) after the transformation $x_1 = x + x_0,\ y_1 = y + y_0$ becomes

$$\frac{dx}{d\tau} = (cx_0 + y_0^2) + cx + (\beta + 2y_0)y + y^2, \quad \frac{dy}{d\tau} = y_0 + y,$$

for which we also have $W = (2 - c)y^6$.

Thus, theorem 5.1 is proved.

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